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Weak symmetry of linear differential operators

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Abstract. The operator equation for the calculation of invariance operators (the weak symmetry operators) connecting the eigenfunctions of one degenerate eigenvalue of a linear operator is formulated. The general properties of invariance operators are described. As an example, the Schrödinger equation with some potentials of interest is considered.

1. Introduction

Consider a linear differential or differential matrix operator $\hat{H}$ and the corresponding eigenvalue and eigenfunction problem

$$\hat{H}\psi_n = E_n\psi_n \tag{1.1}$$

where $n = k, l, m, \ldots$ is the multi-index. Let us suppose that the properties of strong symmetry for the operator $\hat{H}$ are described by the operators $\hat{X}_i$, identically commuting with $\hat{H}$: $[\hat{X}_i, \hat{H}] = \hat{0}$. We shall call the *weak symmetry operators* of $\hat{H}$ a set of such operators $\hat{S}_j$, each of which commutes with $\hat{H}$ on some subset Ω_j of eigenfunctions ψ_n:

$$[\hat{S}_j, \hat{H}]\psi_{n'} = 0 \qquad \psi_{n'} \in \Omega_j. \tag{1.2}$$

Below, the subset Ω_j will be defined in each particular case. The definition of the symmetry and the symmetry operators traditionally used in quantum mechanics (Wigner 1959, Landau and Lifshitz 1974) corresponds to the definition of strong symmetry given above. In mathematical physics (Miller 1977) the notion of commutation on the solutions is used, which is a particular case of the definition of weak symmetry given above: the set Ω is the totality of all degenerate (mutually degenerate) eigenfunctions of the investigated operator $\hat{H}$. The same symmetry is considered in the paper by Poluyanov and Voronin (1983) for the Schrödinger equation.

In this paper, a constructive method for finding the weak symmetry operators is suggested and the usefulness of these operators is shown.

2. Basic operator equation

Let $\hat{Y}_1, \ldots, \hat{Y}_N$ be the set of all mutually commuting strong symmetry operators of $\hat{H}$ and $\lambda_1, \ldots, \lambda_N$ be the corresponding eigenvalues of these operators. Since the eigenfunctions of $\hat{H}$ can be constructed as common eigenfunctions of the operators $\hat{H}, \hat{Y}_1, \ldots, \hat{Y}_N$, we will order the $(N+1)$ indices (discrete or continuum spectral

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quantum numbers) to the eigenfunction and to a multi-index in (1.1) assumed to be $(N+1)$ dimensional: $\psi_n = \psi_{k,k_1,\ldots,k_N}$. Then, finding $\hat{S}$ from an operator equation:

$$[\hat{S}, \hat{H}] = \hat{U}(\hat{H} - E) + \hat{U}_1(\hat{Y}_1 - \lambda_1) + \ldots + \hat{U}_N(\hat{Y}_N - \lambda_N) \tag{2.1}$$

is enough to satisfy the equation $[\hat{S}\hat{H}]\psi_{k,k_1,\ldots,k_N} = 0$. The eigenvalues E; $\lambda_1, \ldots, \lambda_N$ must correspond to the spectral (quantum) numbers $k, k_1, \ldots, k_N$ and $\hat{U}, U_1, \ldots, \hat{U}_N$ are the linear differential (or differential matrix) operators.

The operator equation (2.1) is equivalent to the system of operator equations:

$$\begin{aligned} &[\hat{S}\hat{H}] = 0 \\ &\hat{H} = E \qquad \hat{Y}_1 = \lambda_i \qquad i = 1, \ldots, N \end{aligned} \tag{2.1a}$$

for the weak symmetry operator $\hat{S}$.

The weak symmetry operator $\hat{S}$, obeying the operator equation (2.1), commutes with $\hat{H}$ only on the single eigenfunction (if $\hat{U}, \hat{U}_1, \ldots, \hat{U}_N$ differ from zero). Later, we shall show that in this case the eigenfunction belongs to the degenerate eigenvalue, if ψ_n and $\hat{S}\psi_n$ do not simply differ by the constant phase factor. However, if some operators $\hat{U}, \hat{U}_1, \ldots, \hat{U}_N$ on the right-hand side of (2.1) are the operator zeros, then operator $\hat{S}$ commutes with $\hat{H}$ on the larger set of the eigenfunctions. More exactly, if $\hat{U}_j = \hat{0}$, then an equation $[\hat{S}, \hat{H}]\psi_{\ldots k_j \ldots} = 0$ exists for all permissible values of k_j. If $\hat{U}_j = \hat{U}_i = \hat{0}$, then $[\hat{S}, \hat{H}]\psi_{\ldots k_i \ldots k_j} = 0$ for all the permissible values of k_i, k_j, etc. If $\hat{U} \neq 0$, but $\hat{U}_1, \ldots, \hat{U}_N = 0$, then we have a type of weak symmetry, considered by Miller (1977) and Poluyanov and Voronin (1983). If $\hat{U} = \hat{U}_1 = \ldots = \hat{U}_N = 0$, then all quantum numbers in the equation $[\hat{S}, \hat{H}]\psi_{k,k_1,\ldots,k_N} = 0$ are arbitrary, $\hat{S}$ commutes with $\hat{H}$ on the all the eigenfunctions and due to an assumed completeness of the system of eigenfunctions $[\hat{S}, \hat{H}] \equiv \hat{0}$. In this latter case $\hat{S}$ is a strong symmetry operator ($\hat{S}$ commutes with $\hat{H}$ identically).

Consider an example. Let $\hat{H}$ be a non-relativistic quantum Hamiltonian of a particle in a centrally symmetric field. In this case the components of orbital angular momentum $\hat{\boldsymbol{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$ are strong symmetry operators. We have $\hat{Y}_1 = \hat{L}^2$, $\hat{Y}_2 = \hat{L}_z$ in the capacity of the maximum set of mutually commuting operators. Equation (2.1) takes the form

$$[\hat{S}, \hat{H}] = \hat{U}(\hat{H} - E) + \hat{U}_L[\hat{L}^2 - L(L+1)] + \hat{U}_M(\hat{L}_z - M). \tag{2.2}$$

We shall assume that the energy E belongs to the continuum spectra. An operator $\hat{S}$, defined by equation (2.2) with the full right-hand side, commutes with $\hat{H}$ only on the single eigenfunction and depends on all three quantum numbers

$$[\hat{S}, \hat{H}]\psi_{ELM} = 0 \qquad \hat{S} = \hat{S}_{ELM}.$$

The following seven variants of symmetry arise when the operator of weak symmetry depends on the lesser quantity of quantum numbers:

$$\begin{array}{lll} \hat{U} = \hat{0}(\hat{S}_{LM}) & \hat{U}_L = \hat{0}(\hat{S}_{EM}) & \hat{U}_M = \hat{0}(\hat{S}_{EL}) \\ \hat{U}_L = \hat{U}_M = \hat{0}(\hat{S}_E) & \hat{U} = \hat{U}_M = \hat{0}(\hat{S}_L) & \\ \hat{U} = \hat{U}_L = \hat{0}(\hat{S}_M) & \hat{U} = \hat{U}_L = \hat{U}_M = \hat{0}(\hat{S}). & \end{array}$$

Strong symmetry operators do not depend on quantum numbers. Note that strong symmetry is more important for physical applications than weak symmetry; however, weak symmetry is wider spread. Below we shall consider examples where weak symmetry permits a separation of variables in equation (1.1).

We shall assume that the differential or differential matrix symmetry operators $\hat{S}$ contain the differentiation operations of order 1 or 2. The methods of searching for the symmetry operators $\hat{S}$ from (2.1) are analogous to those used in our previous paper (Poluyanov and Voronin 1983). They will also be explained by considering concrete examples in § 4.

3. General properties of weak symmetry operators

Let us assume that the operator $\hat{S}_i$ is a solution of the operator equation (2.1). As above, designate by means of Ω_i a set of eigenfunctions ψ_n for $\hat{H}$. On this set $\hat{S}_i$ commutes with $\hat{H}$: $\hat{H}\psi_n = E_n\psi_n$, $[\hat{S}_i, \hat{H}]\psi_n = 0$ if $\psi_n \in \Omega_i$. By means of $S_j(\Omega_i)$ we designate a map of set Ω_i as a result of the action of operator $\hat{S}_j$ on the set Ω_i.

Weak symmetry operators have the following properties.

(i) $\hat{S}$ transforms the mutually degenerate eigenfunctions of $\hat{H}$ into one another. Therefore $\hat{S}$ is an invariance operator for the subspace of degenerate levels of $\hat{H}$, i.e. ψ_n and $\hat{S}\psi_n$ are the mutually degenerate eigenfunctions. Actually

$$\hat{H}\hat{S}\psi_n = [\hat{S}\hat{H} - \hat{U}(\hat{H} - E) - \hat{U}_1(\hat{Y}_1 - \lambda_1) - \ldots]\psi_{k,k_1,\ldots,k_N} = \hat{S}\hat{H}\psi_n = E\hat{S}\psi_n.$$

(ii) A sum $\hat{S}_i + \hat{S}_j$ commutes with $\hat{H}$ on the intersection of sets $\Omega_i \cap \Omega_j$. If $\Omega_i \cap \Omega_j = \varnothing$, then $\hat{S}_i + \hat{S}_j$ is not a weak symmetry operator.

(iii) A product $\hat{S}_i\hat{S}_j$ commutes with $\hat{H}$ on the intersection $\Omega_i \cap S_j(\Omega_j)$ and is a weak symmetry operator if $\Omega_i \cap S_j(\Omega_j) \neq \varnothing$.

(iv) A totality of operators $\hat{S}_i$ with the coincided sets Ω_i $(\Omega_1 = \Omega_2 = \ldots)$ forms a linear space.

(v) If $\hat{U}_1 = \hat{U}_2 = \ldots = 0$, then a totality of weak symmetry operators of a given degenerate eigenvalue (Ω_i coincide with each other and consist of all mutually degenerate eigenfunctions of $\hat{H}$) may have a Lie algebra structure. This algebra, generally speaking, is the infinite-dimensional one. We have

$$[\hat{S}', \hat{H}] = \hat{U}'(\hat{H} - E) \qquad [\hat{S}'', \hat{H}] = \hat{U}''(H - E)$$

$$\begin{aligned}[[\hat{S}', \hat{S}''], \hat{H}] &= [\hat{S}'\hat{S}'' - \hat{S}''\hat{S}', \hat{H}] = [\hat{S}'\hat{S}'', \hat{H}] - [\hat{S}''\hat{S}', \hat{H}] \\ &= \hat{S}'[\hat{S}'', \hat{H}] + [\hat{S}', \hat{H}]\hat{S}'' - [\hat{S}'', \hat{H}]\hat{S}' - \hat{S}''[\hat{S}', \hat{H}] \\ &= \hat{S}'\hat{U}''(\hat{H} - E) + \hat{U}'(\hat{H} - E)\hat{S}'' - \hat{U}''(\hat{H} - E)\hat{S}' - \hat{S}''\hat{U}'(\hat{H} - E) \\ &= \{[\hat{S}', \hat{U}''] + [\hat{U}', \hat{S}''] + [\hat{U}'', \hat{U}']\}(\hat{H} - E)\end{aligned}$$

which is a commutator $[\hat{S}', \hat{S}'']$ and is also a weak symmetry operator of a given degenerate eigenvalue $\hat{H}$.

Let a Hermitian weak symmetry operator induce a point transformation of variables (Shapovalov 1969, Miller 1977). Then we have the following properties.

(vi) The point transformation of variables of the weak symmetry operator $\hat{S}$, generally speaking, does not transform one solution of (1.1) into another (that is, it does not generate the solutions) and does not retain the initial equation $(H - E)\psi = 0$ invariant. The infinitesimal point transformation of variables in the linear approximation (the small group parameters) generates the solutions and retains the initial equation (1.1) invariant.

Let us introduce the operator of the point transformation group $T = \exp(\mathrm{i}t\hat{S})$, where t is a real group parameter. Apparently, an expression $\psi_t = \hat{T}\psi = \exp(\mathrm{i}t\hat{S})\psi$ is not a solution of the initial equation, because the powers $\hat{S}$ in the general case are not operators of weak symmetry.

The equation $(\hat{H}-E)\psi=0$ in new variables takes the form

$$e^{it\hat{S}}(\hat{H}-E)\,e^{-it\hat{S}}\psi_t=0.$$

The new equation does not coincide with the previous equation, because the operator

$$e^{it\hat{S}}(\hat{H}-E)e^{-it\hat{S}}=\hat{H}-E+it[\hat{S},\hat{H}]-\tfrac{1}{2}t^2[\hat{S},[\hat{S},\hat{H}]]+\ldots$$

is not proportional to $\hat{H}-E$ on the subset of eigenfunctions Ω (such that $[\hat{S},\hat{H}]_\Omega=\hat{0}$). If t is small, then in the linear approximation we have

$$\psi_t\simeq(1+it\hat{S})\psi$$

and ψ_t is the solution of the initial equation. The function ψ_t is the linear combination of the mutually degenerate eigenfunctions ψ_n, since in the same approximation, the expression

$$\begin{aligned}e^{it\hat{S}}(\hat{H}-E)e^{-it\hat{S}}\\ &\approx\hat{H}-E+it[\hat{U}(\hat{H}-E)+\hat{U}_1(\hat{Y}_1-\lambda_1)+\ldots+\hat{U}_N(\hat{Y}_N-\lambda_N)]\\ &=(1+it\hat{U})(\hat{H}-E)\end{aligned}$$

exists on the set Ω.

Thus, $(\hat{H}-E)\psi_t\simeq0$, i.e. the approximate invariance of the equation on the set Ω exists.

(vii) If $\hat{U}_1=\hat{U}_2=\ldots=\hat{U}_N=0$ in (2.1), then a generator of the point transformation group $\hat{T}$ generates the solutions of the initial equation (1.1). In this case the initial equation is invariant concerning $\hat{T}$, because

$$e^{it\hat{S}}(\hat{H}-E)e^{-it\hat{S}}=\hat{U}_t(\hat{H}-E)$$

where $\hat{U}_t$ is some proportional operator, depending on the group parameter t. Thus, weak symmetry of the type

$$[\hat{S},\hat{H}]=\hat{U}(\hat{H}-E)$$

appears by its properties to be the closest to the strong symmetry

$$[\hat{S},\hat{H}]\equiv\hat{0}.$$

To conclude this section we formulate the physical interpretation of the weak symmetry. It may be interpreted as the dynamical symmetry of the system, reduced on the restricted types of motion (for the case when $\hat{H}$ is the Hamiltonian of the system). In the general case this symmetry does not possess group or algebraic properties. Nevertheless, its importance is shown in the following section. The weak symmetry is more widespread than strong symmetry. The weak symmetry may be a manifestation of strong symmetry on the limited class of motions. In particular, the weak symmetry operators may correspond to the discrete transformations of symmetry. This will be illustrated by the following examples.

4. The weak symmetry for quantum particle motion in the axial symmetrical and centrally symmetrical potentials

4.1. *Two-dimensional axially symmetrical potentials*

The Hamiltonian of the system in polar coordinates has the following form:

$$\hat{H} = -\frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) + V(r). \tag{4.1}$$

Consider the differential weak symmetry operators, which are the solutions of the operator equation

$$[\hat{S}, \hat{H}] = \hat{W}(\hat{L}_z - m) \tag{4.2}$$

where $\hat{L}_z = -\mathrm{i}\partial/\partial\varphi$ is the strong symmetry operator of $\hat{H}$ (4.1). This is the projection operator of orbital angular momentum on the symmetry axis of the potential and m is the eigenvalue of $\hat{L}_z$. The weak symmetry operators, defined by the operator equation (4.2), depend on one quantum number m. Thus, each operator of this type may be applied to the wavefunction with the given value of quantum number m at any value of the energy E.

We carry out the analysis in the class of first-order differential symmetry operators

$$\hat{S} = B_1(r, \varphi)\frac{\partial}{\partial r} + B_2(r, \varphi)\frac{\partial}{\partial \varphi} + C(r, \varphi). \tag{4.3}$$

In this case the commutation $[\hat{S}, \hat{H}]$ is a second-order differential operator and therefore the operator $\hat{W}$ in (4.2) must be a first-order differential operator:

$$\hat{W} = u_1(r, \varphi)\frac{\partial}{\partial r} + u_2(r, \varphi)\frac{\partial}{\partial \varphi} + u_3(r, \varphi). \tag{4.4}$$

Substituting (4.1), (4.3) and (4.4) into (4.2) and equating coefficients of the same differential operators in the left- and right-hand sides we obtain the following system of linear partial differential equations:

$$\frac{\partial B_1}{\partial r} = 0 \tag{4.5a}$$

$$\frac{B_1}{r^3} + \frac{1}{r^2}\frac{\partial B_2}{\partial \varphi} = -\mathrm{i}u_2 \tag{4.5b}$$

$$\frac{\partial B_2}{\partial r} + \frac{1}{r^2}\frac{\partial B_1}{\partial \varphi} = -\mathrm{i}u_1 \tag{4.5c}$$

$$\frac{1}{2}\left(\Delta B_1 + \frac{B_1}{r^2}\right) + \frac{\partial C}{\partial r} = -mu_1 \tag{4.5d}$$

$$\tfrac{1}{2}\Delta B_2 + \frac{1}{r^2}\frac{\partial C}{\partial \varphi} = -mu_2 - \mathrm{i}u_3 \tag{4.5e}$$

$$\tfrac{1}{2}\Delta C + B_1\frac{\mathrm{d}V}{\mathrm{d}r} = -mu_3 \qquad \Delta = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}. \tag{4.5f}$$

Excluding the functions u_1, u_2, u_3 from the system (4.5) we obtain

$$\frac{\partial B_1}{\partial r}=0 \tag{4.6a}$$

$$\frac{1}{2}\left(\Delta B_1+\frac{B_1}{r^2}\right)+\frac{\mathrm{i}m}{r^2}\frac{\partial B_1}{\partial\varphi}+\frac{\partial A}{\partial r}=0 \tag{4.6b}$$

$$\tfrac{1}{2}\Delta A+\frac{\mathrm{i}m}{r^2}\frac{\partial A}{\partial\varphi}+B_1\left(\frac{\mathrm{d}V}{\mathrm{d}r}-\frac{m^2}{r^3}\right)=0 \tag{4.6c}$$

$$A=\mathrm{i}mB_2+C. \tag{4.6d}$$

The case $A=B_1=0$ corresponds to the trivial symmetry operators

$$\hat{S}_0^{(m)}=\mathrm{i}B_2(r,\varphi)(\hat{L}_z-m) \tag{4.7}$$

which any solution $\psi=f(r)\mathrm{e}^{\mathrm{i}m\varphi}$ of the Schrödinger equation with the Hamiltonian (4.1) reduces to zero. The operator $\hat{S}_0^{(m)}$ may always be added to any other symmetry operator $\hat{S}^{(m)}$.

If $B_1=0$, $A=A(\varphi)$, then we obtain the following weak symmetry operator:

$$\hat{S}_*^{(m)}=\mathrm{e}^{-2\mathrm{i}m\varphi}. \tag{4.8}$$

This multiplication operator on the function transforms a solution with quantum number m into a solution with quantum number $(-m)$. Weak symmetry of this type is the 'track' of strong discrete symmetry (a reflection in the plane containing the symmetry axis of the potential). This last is represented by the operator (4.3), which is defined on the subset of functions with a given m. If $2V(r)\neq\alpha/r+\beta/r^2$, then $\hat{S}_*^{(m)}$ is the single symmetry operator in the class (4.3).

4.1.1. The Kratzer potential $V(r)$. If $V(r)$ is the two-dimensional Kratzer potential

$$2V(r)=\alpha/r+\beta/r^2 \tag{4.9}$$

(α and β are real constants), then from the system of equations (4.6) it follows that

$$B_1=B_1(\varphi)\qquad A(r,\varphi)=\frac{1}{r}I(\varphi)+K(\varphi) \tag{4.10}$$

$$2I(\varphi)=B_1''(\varphi)+2\mathrm{i}mB_1'(\varphi)+B_1(\varphi) \tag{4.11a}$$

$$\alpha B_1(\varphi)=K''(\varphi)+2\mathrm{i}mK'(\varphi) \tag{4.11b}$$

$$2(m^2+\beta)B_1=I''(\varphi)+2\mathrm{i}mI'(\varphi)+I(\varphi). \tag{4.11c}$$

Using equations (4.11) we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}+2\mathrm{i}m\frac{\mathrm{d}}{\mathrm{d}\varphi}+1+2(m^2+\beta)^{1/2}\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}+2\mathrm{i}m\frac{\mathrm{d}}{\mathrm{d}\varphi}+1-2(m^2+\beta)^{1/2}\right)B_1=0. \tag{4.12}$$

The general solution of equation (4.12) has the following form:

$$B_1(\varphi)=-C_1\mathrm{e}^{\mathrm{i}\lambda_1\varphi}-C_2\mathrm{e}^{\mathrm{i}\lambda_2\varphi}+C_3\mathrm{e}^{\mathrm{i}\lambda_3\varphi}+C_4\mathrm{e}^{\mathrm{i}\lambda_4\varphi} \tag{4.13}$$

where

$$\lambda_{1,2}=-m\pm[1+m^2+2(m^2+\beta)^{1/2}]^{1/2}$$
$$\lambda_{3,4}=-m\pm[1+m^2-2(m^2+\beta)^{1/2}]^{1/2} \qquad C_i=\text{constant}. \tag{4.14}$$

From (4.11a) and (4.11b) we obtain

$$I(\varphi)=(m^2+\beta)^{1/2}(C_1e^{i\lambda_1\varphi}+C_2e^{i\lambda_2\varphi}+C_3e^{i\lambda_3\varphi}+C_4e^{i\lambda_4\varphi}) \tag{4.15}$$

$$K(\varphi)=\alpha\frac{C_1e^{i\lambda_1\varphi}+C_2e^{i\lambda_2\varphi}}{2(m^2+\beta)^{1/2}+1}+\alpha\frac{C_3e^{i\lambda_3\varphi}+C_4e^{i\lambda_4\varphi}}{2(m^2+\beta)^{1/2}-1}. \tag{4.16}$$

Suppose that in equations (4.15) and (4.16) the following inequalities are fulfilled:

$$m^2+\beta\neq 0 \qquad 2(m^2+\beta)^{1/2}\pm 1\neq 0. \tag{4.17}$$

Thus, the potential (4.9) permits four operators of weak symmetry:

$$\hat{S}_1^{(m)}=e^{i\lambda_1\varphi}\left(\frac{\partial}{\partial r}-\frac{1}{r}(m^2+\beta)^{1/2}-\frac{\alpha}{2(m^2+\beta)^{1/2}+1}\right) \tag{4.18a}$$

$$\hat{S}_2^{(m)}=e^{i\lambda_2\varphi}\left(\frac{\partial}{\partial r}-\frac{1}{r}(m^2+\beta)^{1/2}-\frac{\alpha}{2(m^2+\beta)^{1/2}+1}\right) \tag{4.18b}$$

$$\hat{S}_3^{(m)}=e^{i\lambda_3\varphi}\left(\frac{\partial}{\partial r}+\frac{1}{r}(m^2+\beta)^{1/2}+\frac{\alpha}{2(m^2+\beta)^{1/2}-1}\right) \tag{4.18c}$$

$$\hat{S}_4^{(m)}=e^{i\lambda_4\varphi}\left(\frac{\partial}{\partial r}+\frac{1}{r}(m^2+\beta)^{1/2}+\frac{\alpha}{2(m^2+\beta)^{1/2}-1}\right). \tag{4.18d}$$

Note that the weak symmetry operators 1, $\hat{S}_*^{(m)}$, $\hat{S}_i^{(m)}$ ($i=1,2,3,4$) may be considered as a linear space basis: an arbitrary linear combination of these operators is the weak symmetry operator on the set of eigenfunctions which correspond to the arbitrary energy and the fixed quantum number m. The main property of these operators lies in the connection of mutually degenerate eigenfunctions of $\hat{H}$. However, it is impossible to introduce the structure of Lie algebras in the above mentioned linear space, since the product and commutators of these operators are not weak symmetry operators.

The functions $\hat{S}_i^{(m)}f(r)\exp(im\varphi)$ are single-valued only when λ_i is an integer number. If $\beta=0$, then all four operators $\hat{S}_i^{(m)}$ obey this condition. If

$$\beta=\tfrac{1}{4}(l^2-m^2-1)^2-m^2 \tag{4.19}$$

(l is a whole number), then λ_1, λ_2 are integer numbers at $l^2-m^2-1>0$ and λ_3, λ_4 are integer numbers at $l^2-m^2-1<0$. If β has the form (4.19), then all eigenvalues E_n are characterised for the given m by additional degeneration, which is not connected with the substitution $m\to -m$.

For the case $\beta=0$, i.e. for the two-dimensional Coulomb potential, the operators $\hat{S}_1^{(m)}$, $\hat{S}_4^{(m)}$ coincide with the linear combination of strong symmetry operators $\hat{A}_1$, $\hat{A}_2$, which are the components of the two-dimensional Runge-Lentz vector (reduced on the subset of functions with the definite quantum number m):

$$\hat{A}_1=x\frac{\partial^2}{\partial y^2}-y\frac{\partial^2}{\partial x\partial y}-\frac{1}{2}\frac{\partial}{\partial x}-\frac{\alpha x}{2(x^2+y^2)^{1/2}}$$
$$\to(im\sin\varphi+\tfrac{1}{2}\cos\varphi)\frac{\partial}{\partial r}-(m^2\cos\varphi+\tfrac{1}{2}im\sin\varphi)\frac{1}{r}-\tfrac{1}{2}\alpha\cos\varphi$$

$$\hat{A}_2 = y\frac{\partial^2}{\partial x^2} - x\frac{\partial^2}{\partial x\partial y} - \frac{1}{2}\frac{\partial}{\partial y} - \frac{\alpha y}{2(x^2+y^2)^{1/2}}$$

$$\rightarrow (-\mathrm{i}m\cos\varphi + \tfrac{1}{2}\sin\varphi)\frac{\partial}{\partial r} - (m^2\sin\varphi - \tfrac{1}{2}\mathrm{i}m\cos\varphi)\frac{1}{r} - \tfrac{1}{2}\alpha\sin\varphi \qquad (4.20)$$

$$\hat{S}_1^{(m)} = \frac{\hat{A}_1 + \mathrm{i}\hat{A}_2}{m+\frac{1}{2}} \qquad \hat{S}_4^{(m)} = \frac{\hat{A}_1 - \mathrm{i}\hat{A}_2}{-m+\frac{1}{2}} \qquad (\alpha < 0).$$

They transform the Coulomb wavefunction $\psi_{n_r}^m$ in the following way:

$$\hat{S}_1^{(m)}\psi_{n_r}^m \rightarrow \psi_{n_r-1}^{m+1} \qquad \hat{S}_4^{(m)}\psi_{n_r}^m \rightarrow \psi_{n_r+1}^{m-1} \qquad (4.21)$$

where

$$\psi_{n_r}^m = Nr^{|m|}\mathrm{e}^{-\gamma r}\Phi(-n_r, 2|m|+1, 2\gamma r)\mathrm{e}^{\mathrm{i}m\varphi} \qquad \gamma = (-2E_n)^{1/2}$$

where n_r is the radial quantum number and $\Phi(.,.,.)$ is a confluent hypergeometric function. Thus, the operators $\hat{S}_1^{(m)}$, $\hat{S}_4^{(m)}$ change quantum numbers $n_r \rightarrow n_r \mp 1$, $m \rightarrow m \pm 1$ and keep $E_n = -\alpha^2/8(n_r + m + 1)^2$ invariable. As follows from (4.20), the form of the reduced strong symmetry operators $\hat{A}_1, \hat{A}_2$ ($\hat{S}_1^{(m)}$, $\hat{S}_4^{(m)}$) becomes simplified. They break up into the product of the radial and angular factors and, moreover, the radial factor is a first-order differential operator. The classical analogues of $\hat{S}_1^{(m)}$, $\hat{S}_4^{(m)}$ also become simplified. This simplification is connected with the reduction of the phase space, which is accessible to the motion at the fixed motion integral. The analogous conclusions are also correct for the reduced strong symmetry in the following examples.

We note that the Coulomb radial Hamiltonian

$$\hat{H}_m = -\frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{m^2}{r^2}\right) + \frac{\alpha}{2r}$$

is factorised in the following form:

$$\hat{H}_m = -\tfrac{1}{2}\hat{S}_1^{m-1}\hat{S}_4^m - \frac{(\alpha/2)^2}{2(m-\frac{1}{2})^2} = -\tfrac{1}{2}\hat{S}_4^{m+1}\hat{S}_1^m - \frac{(\alpha/2)^2}{2(m+\frac{1}{2})^2}$$

where $\hat{S}_1^m$, $\hat{S}_4^m$ are the radial factors of the weak symmetry operators $\hat{S}_1^{(m)}$, $\hat{S}_4^{(m)}$. Thus, the radial operators $\hat{S}_1^{m-1}$, $\hat{S}_4^m$ ($\hat{S}_4^{m+1}$, $\hat{S}_1^m$) for the Coulomb radial Hamiltonian are the same operators that one should be able to find by the factorisation method (Infeld and Hull 1951).

4.1.2. The two-dimensional oscillator $V = \frac{1}{2}\lambda^2(x^2+y^2)$. The solution of the operator equation

$$[\hat{S}, \hat{H}] = \hat{U}(\hat{H} - E_n) + \hat{U}_1(\hat{L}_z - m) \qquad (4.22)$$

gives the following weak symmetry operators, which depend on the energy E_n and the quantum number m:

$$\hat{S}_1 = \mathrm{e}^{2\mathrm{i}\varphi}\left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{m}{r^2} + \frac{E_n}{m+1}\right) = \hat{S}_1^{(m,n)}$$

$$\hat{S}_2 = \mathrm{e}^{-2\mathrm{i}\varphi}\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{m}{r^2} - \frac{E_n}{m-1}\right) = \hat{S}_2^{(m,n)} \qquad (4.23)$$

$$E_n = \lambda(2n_r + m + 1) = \lambda(n+1).$$

The result of the action of $\hat{S}_1$ and $\hat{S}_2$ on the eigenfunction of $\hat{H}$ at the definite energy and momentum values has the form

$$\hat{S}_1\psi^m_{n_r} \to \psi^{m+2}_{n_r-1} \qquad \hat{S}_2\psi^m_{n_r} \to \psi^{m-2}_{n_r+1} \tag{4.24}$$

where $\psi^m_{n_r} = \text{constant} \times r^{|m|} \exp(-\lambda r^2/2)\Phi(-n_r, 1+|m|, \lambda r^2) \exp(im\varphi)$. Also, $\hat{S}_1$ and $\hat{S}_2$ are the linear combination of the reduced strong symmetry operators $\hat{M}_1$ and $\hat{M}_2$:

$$\begin{aligned} \hat{M}_1 &= xy - \frac{\partial^2}{\partial x \partial y} = \frac{i}{2}(m+1)\hat{S}_1 + \frac{i}{2}(m-1)\hat{S}_2 \\ \hat{M}_2 &= x^2 - y^2 - \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -\frac{m+1}{2}\hat{S}_1 + \frac{m-1}{2}\hat{S}_2. \end{aligned} \tag{4.25}$$

4.2. *Three-dimensional centrally symmetrical potentials*

4.2.1. The Coulomb potential $V(r) = +\alpha/2r$. In this case the weak symmetry operators, which are the solution of the operator equation

$$[\hat{S}, \hat{H}] = \hat{U}_1[\hat{L}^2 - l(l+1)] \tag{4.26}$$

have the form

$$\begin{aligned} \hat{S}_1 &= \left(\frac{d}{dr} - \frac{l}{r} - \frac{\alpha}{2(l+1)}\right)\left((1-\cos^2\theta)\frac{d}{d\cos\theta} - (l+1)\cos\theta\right) \\ \hat{S}_2 &= \left(\frac{d}{dr} + \frac{l+1}{r} + \frac{\alpha}{2l}\right)\left((1-\cos^2\theta)\frac{d}{d\cos\theta} + l\cos\theta\right) \end{aligned} \tag{4.27}$$

where $\hat{L}$ is the orbital angular momentum operator. Acting on the Coulomb wavefunction, the operators $\hat{S}_1$ and $\hat{S}_2$ change the quantum numbers in the following way: $n_r \to n_r \mp 1$, $l \to l \pm 1$, $m \to m$, and leave the energy E invariant. The operators changing the quantum number m may be determined in the same way. The operator equation (4.26) may also be solved for the Kratzer potential $2V(r) = \beta/r^2 + \alpha/r$.

4.2.2. The three-dimensional harmonic potential $V(r) = \frac{1}{2}\lambda r^2$. The weak symmetry operators for this potential have the form

$$\begin{aligned} \hat{S}_1 &= \left(\frac{1}{r}\frac{d}{dr} - \frac{l}{r^2} + \frac{E_n}{l+\frac{3}{2}}\right)\left((1-\cos^2\theta)\frac{d}{d\cos\theta} - (l+1)\cos\theta\right) \\ \hat{S}_2 &= \left(\frac{1}{r}\frac{d}{dr} + \frac{l-1}{r^2} - \frac{E_n}{l-\frac{1}{2}}\right)\left((1-\cos^2\theta)\frac{d}{d\cos\theta} + l\cos\theta\right). \end{aligned} \tag{4.28}$$

They are the solutions of the operator equation

$$[\hat{S}, \hat{H}] = \hat{U}(\hat{H} - E_n) + \hat{U}_1[\hat{L}^2 - l(l+1)]. \tag{4.29}$$

Acting on the wavefunction at the definite energy E_n and the orbital angular momentum l values, $\hat{S}_1$ and $\hat{S}_2$ change the quantum numbers in the following way: $n_r \to n_r \mp 1$, $l \to l \pm 2$, $m \to m$ and also leave E_n invariant.

Note that the weak symmetry operators are not always reduced strong symmetry operators (an example is the case of the weak symmetry operators for the two-dimensional Kratzer potential).

So, for the spherically symmetric or axially symmetric potentials, the weak symmetry operator $\hat{S}$ may be factorised into the product of radial and angular operators. For the Coulomb and oscillator potentials the action of these radial and angular operators is analogous to the action of corresponding operators of dynamical non-invariance algebra. However, $\hat{S}$ is the invariance operator. In the general case the weak symmetry operators $\hat{S}_j$ depend on the quantum numbers and do not form any Lie algebra.

5. The symmetry of one class of differential matrix Schrödinger equation

Consider a differential matrix Schrödinger equation of the form

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\begin{pmatrix}0 & F(x+\mathrm{i}y)\\ \bar{F}(x-\mathrm{i}y) & 0\end{pmatrix}\right]\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}=E\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \qquad (5.1)$$

where $F(x+\mathrm{i}y)$ is an arbitrary analytical function of a variable $z=x+\mathrm{i}y$. Equation (5.1) at some concrete functions $F(x+\mathrm{i}y)$ describes the nuclear dynamics in the neighbourhood of the point of two-fold degeneracy for the potential energy surfaces. We enumerate those cases when $F(z)$ corresponds to the known Hamiltonians of the following physical systems.

(i) $F(z)=\alpha z$ is a conical intersection of potential surfaces.

(ii) $F(z)=\beta z^2$, γz^4 is a parabolical intersection of potential surfaces.

(iii) $F(z)=\delta/z$ is a particle with spin $\frac{1}{2}$ in a magnetic field of linear current.

The Hamiltonian (5.1) may be written as

$$\hat{H}=-2\frac{\partial^2}{\partial z\partial\bar{z}}+F(z)\hat{\sigma}_{12}+\bar{F}(\bar{z})\hat{\sigma}_{21} \qquad (5.2)$$

where

$$\bar{z}=x-\mathrm{i}y$$

$$\hat{\sigma}_{12}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \qquad \hat{\sigma}_{21}=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

We shall find the symmetry operators $\hat{S}$ of the Hamiltonian $\hat{H}$ (5.2), which commute with the $\hat{H}$ on a set of solutions for equation (5.1) with zero energy, i.e.

$$[\hat{S},\hat{H}]=\hat{U}\hat{H}.$$

Let us carry out an analysis in the class of second-order operators

$$\hat{S}=A_1\frac{\partial^2}{\partial z^2}+2A_{12}\frac{\partial^2}{\partial z\partial\bar{z}}+A_2\frac{\partial^2}{\partial\bar{z}^2}+\hat{B}_1\frac{\partial}{\partial z}+\hat{B}_2\frac{\partial}{\partial\bar{z}}+\hat{C} \qquad (5.3)$$

where $\hat{B}_1$, $\hat{B}_2$ are the functional matrix, $\hat{C}$ is the constant matrix and $A_{1,2}$, A_{12} are functions of z, $\bar{z}$. As in the considered case, the ideal trivial symmetry (Miller 1977) operators have the form

$$\hat{S}_0=\Phi(z,\bar{z})\left(-2\frac{\partial^2}{\partial z\partial\bar{z}}+F(z)\hat{\sigma}_{12}+\bar{F}(\bar{z})\hat{\sigma}_{21}\right)$$

where the coefficient A_{12} may always be reduced to zero by the similarity transformation with the corresponding choice of the function $\Phi(z,\bar{z})$. Thus, we consider $A_{12}=0$ without loss of generality and determine the symmetry operators in the form

$$\hat{S}=A_1\frac{\partial^2}{\partial z^2}+A_2\frac{\partial^2}{\partial\bar{z}^2}+\hat{B}_1\frac{\partial}{\partial z}+\hat{B}_2\frac{\partial}{\partial\bar{z}}+\hat{C}. \qquad (5.3a)$$

The coefficients $\hat{B}_1$, $\hat{B}_2$ and $\hat{C}$ are considered to be diagonal matrices. The proportionality operator $\hat{U}$ should be the differential matrix first-order operator, since $[\hat{S}, \hat{H}]$ is the differential matrix third-order operator. Suppose

$$\hat{U} = a_1 \frac{\partial}{\partial z} + a_2 \frac{\partial}{\partial \bar{z}} + \hat{b} \tag{5.4}$$

where a_1, a_2 are functions of z and $\bar{z}$ and $\hat{b}$ is a diagonal functional matrix. The assumption of the diagonality of the $\hat{B}_1$, $\hat{B}_2$, $\hat{C}$ and $\hat{b}$ matrices and also the particular form $\hat{U}$ are made in order to simplify the calculations. The following calculations show that under these limitations the analysis does not lose its richness of content.

Calculating the commutator $[\hat{S}, \hat{H}]$ and the product $\hat{U}\hat{H}$ and equating the coefficients of the same operators in the left- and right-hand sides of the operator equation $[\hat{S}, \hat{H}] = \hat{U}\hat{H}$ results in the following system of equations:

$$\partial A_1/\partial \bar{z} = 0 \qquad a_1 = -\mathrm{d}A_1/\mathrm{d}z = a_1(z) \tag{5.5a}$$

$$\partial A_2/\partial z = 0 \qquad a_2 = -\mathrm{d}A_2/\mathrm{d}\bar{z} = a_2(\bar{z}) \tag{5.5b}$$

$$\partial \hat{B}_1/\partial \bar{z} = 0 \qquad \partial \hat{B}_2/\partial z = 0 \qquad \hat{b} = -(\mathrm{d}\hat{B}_1/\mathrm{d}z) - (\mathrm{d}\hat{B}_2/\mathrm{d}\bar{z}) \tag{5.5c}$$

$$2A_1(\mathrm{d}F/\mathrm{d}z)\hat{\sigma}_{12} + F[\hat{B}_1, \hat{\sigma}_{12}] + \bar{F}[\hat{B}_1, \hat{\sigma}_{21}] = a_1(F\hat{\sigma}_{12} + \bar{F}\hat{\sigma}_{21}) \tag{5.5d}$$

$$2A_2(\mathrm{d}\bar{F}/\mathrm{d}\bar{z})\hat{\sigma}_{21} + F[\hat{B}_2, \hat{\sigma}_{12}] + \bar{F}[\hat{B}_2, \hat{\sigma}_{21}] = a_2(F\hat{\sigma}_{12} + \bar{F}\hat{\sigma}_{21}) \tag{5.5e}$$

$$\begin{aligned} A_1 \frac{\mathrm{d}^2 F}{\mathrm{d}z^2} \hat{\sigma}_{12} + A_2 \frac{\mathrm{d}^2 \bar{F}}{\mathrm{d}\bar{z}^2} \hat{\sigma}_{21} + \hat{B}_1 \hat{\sigma}_{12} \frac{\mathrm{d}F}{\mathrm{d}z} + \hat{B}_2 \hat{\sigma}_{21} \frac{\mathrm{d}\bar{F}}{\mathrm{d}\bar{z}} + F[\hat{C}, \hat{\sigma}_{12}] + \bar{F}[\hat{C}, \hat{\sigma}_{21}] \\ = a_1 \frac{\mathrm{d}F}{\mathrm{d}z} \hat{\sigma}_{12} + F\hat{b}\hat{\sigma}_{12} + a_2 \frac{\mathrm{d}\bar{F}}{\mathrm{d}\bar{z}} \hat{\sigma}_{21} + \bar{F}\hat{b}\hat{\sigma}_{21}. \end{aligned} \tag{5.5f}$$

Solving these equations, we obtain the operators

$$\hat{S}_1 = \frac{1}{F} \frac{\partial^2}{\partial z^2} - \frac{1}{F^2} \frac{\mathrm{d}F}{\mathrm{d}z} \hat{\sigma}_{11} \frac{\partial}{\partial z} \qquad (F = F(z)) \tag{5.6}$$

$$\hat{S}_2 = \frac{1}{\bar{F}} \frac{\partial^2}{\partial \bar{z}^2} - \frac{1}{\bar{F}^2} \frac{\mathrm{d}\bar{F}}{\mathrm{d}\bar{z}} \hat{\sigma}_{22} \frac{\partial}{\partial \bar{z}} \qquad (\bar{F} = \bar{F}(\bar{z})) \tag{5.7}$$

which are the symmetry operators of the Hamiltonian (5.2) with an arbitrary function $F(z)$. The important property of $\hat{S}_1$ and $\hat{S}_2$ is their commutativity

$$[\hat{S}_1, \hat{S}_2] = 0. \tag{5.8}$$

If $F(z)$ is arbitrary, then the symmetry of the Hamiltonian is exhausted by weak symmetry operators $\hat{S}_1$ and $\hat{S}_2$ in the class of operators (5.3*a*). In the case of the power function

$$F(z) = \alpha z^s \tag{5.9}$$

where α and s are the complex and real parameters, correspondingly there exists the third symmetry operator. It is a strong symmetry operator

$$\hat{S}_3 = z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} - \frac{s}{2} (\hat{\sigma}_{11} - \hat{\sigma}_{22}). \tag{5.10}$$

This symmetry operator has the meaning of the projection of a total momentum (orbital plus 'spin' momentum) on the symmetry axis of eigenvalues for the potential matrix

$$U_{1,2} = \pm|F(z)| = \pm|\alpha|r^s \qquad r = (x^2+y^2)^{1/2}. \tag{5.11}$$

The symmetry operator $\hat{S}_3$ satisfies the following commutation relations:

$$[\hat{S}_1, \hat{S}_3] = (s+2)\hat{S}_1 \qquad [\hat{S}_2, \hat{S}_3] = -(s+2)\hat{S}_2. \tag{5.12}$$

Taking into account the commutation relations (5.8) and (5.12) we obtain that the symmetry operators $\hat{S}_1$, $\hat{S}_2$ and $\hat{S}_3$ form the basis of a Lie algebra for the Euclidean motion group E(2), if $s+2 \neq 0$. If $F(z)$ has the form (5.9), then the separation of variables can be made in the system of equations (5.1) in two types of coordinates: in the polar coordinates (r, φ) and in the coordinates $(z, \bar{z})$. If $F(z)$ is arbitrary, then the adiabatic potential surfaces $U_{1,2} = \pm|F(z)|$ do not have any geometrical symmetry. In this case the separation of variables can still be made for equations (5.1) in coordinates $(z, \bar{z})$. It is a result of weak symmetry, represented by operators $\hat{S}_1$ and $\hat{S}_2$. Suppose in (5.1)

$$\chi_1 = A_1(z)B_1(\bar{z}) \qquad \chi_2 = A_2(z)B_2(\bar{z}) \tag{5.13}$$

we obtain the second-order differential equations for $A_2(z)$ and $B_1(\bar{z})$

$$(\mathrm{d}^2A_2/\mathrm{d}z^2) - \lambda_1\lambda_2F(z)A_2 = 0 \tag{5.14}$$

$$(\mathrm{d}^2B_1/\mathrm{d}\bar{z}^2) - (4\lambda_1\lambda_2)^{-1}\bar{F}(\bar{z})B_1 = 0 \tag{5.15}$$

where λ_1 and λ_2 are the separation constants. As the operators $\hat{H}$, $\hat{S}_1$ and $\hat{S}_2$ are mutually commuted, they have (at zero energy) common eigenfunctions. If the eigenfunctions of $\hat{S}_1$ and $\hat{S}_2$ have the form (5.13)

$$\hat{S}_{1,2}\begin{pmatrix} A_1B_1 \\ A_2B_2 \end{pmatrix} = \sigma_{1,2}\begin{pmatrix} A_1B_1 \\ A_2B_2 \end{pmatrix}$$

then, after separation of variables, we obtain the same equations (5.14) and (5.15), where σ_1 and σ_2 appear instead of $\lambda_1\lambda_2$ and $1/4\lambda_1\lambda_2$. Thus, $\hat{S}_1$ and $\hat{S}_2$ were actually the operators for the constants of the separation of variables. This fact proves the existence of coupling between the given separation of variables and weak symmetry, represented by the operators $\hat{S}_1$ and $\hat{S}_2$. It should be noted that for the case $s=-1$ the operators $\hat{S}_1$ and $\hat{S}_2$ are equivalent (on the ideal of trivial symmetry $\hat{S}_0$) to the strong symmetry operators

$$\hat{S}'_1 = \hat{S}_1 + (\bar{z}/2\alpha)\hat{H} \qquad [\hat{S}'_1, \hat{H}] = \hat{0}$$

$$\hat{S}'_2 = \hat{S}_2 + (z/2\alpha)\hat{H} \qquad [\hat{S}'_2, \hat{H}] = \hat{0}.$$

They are the matrix analogues of components for the two-dimensional Runge–Lentz vector (Pron'ko and Stroganov 1977). The operators $\hat{S}'_1$ and $\hat{S}'_2$ are the strong symmetry operators of the Hamiltonian (5.2) at $F(z) = \alpha/z$. They form with the $\hat{S}_3$ (on the eigenfunctions of the Hamiltonian) the algebras SO(3) and SO(2, 1) for the discrete and continuum spectra, correspondingly.

6. Conclusion

The analysis of the strong symmetry should precede the necessity of the solution of the basic operator equation (2.1). The functional independent symmetry operators $\hat{X}_i$, identically commuting with the Hamiltonian, should be found; a maximal set of mutually commuting strong symmetry operators $\hat{Y}_i$ must be constructed and the spectrum of all operators $\hat{Y}_i$ should be determined. However, all information on strong symmetry for the Hamiltonians of physical systems is usually known; therefore the operator equation for the study of weak symmetry may be formulated easily.

The use of the operator equation $[\hat{S}, \hat{H}] = \hat{U}(\hat{H} - E)$ and the symmetry operators $\hat{S}$, inducing the point transformation of variables, is equivalent to the analysis of symmetry by the Lie-Ovsjannikov method (Ovsjannikov 1978). In this case the initial equation is invariant with respect to the point transformations. If the weak symmetry operator satisfies the operator equation (2.1) with some $\hat{U}_1, \ldots, \hat{U}_N$ not all equal to zero, then an approximate invariance of equation (1.1) exists only for the infinitesimal transformations, on the definite subset of eigenfunctions $\hat{H}$. The absence of algebraic structure in the linear space of weak symmetry operators is connected with this fact. It is evident that the most significant form of weak symmetry corresponds to the case $\hat{U}_i = 0$ ($i = 1, \ldots, N$). However, the weak symmetry operators, corresponding to some $\hat{U}_1, \ldots, \hat{U}_N$ not all equal to zero, are of interest. The weak symmetry of this type sometimes gives new symmetry operators. This weak symmetry is connected, for some potentials, with the factorisation method. Finally, the weak symmetry operators have an evident interpretation in classical mechanics: they are integrals of motion on reduced phase space.

At any $\hat{U}_i \neq \hat{0}$ the weak symmetry operators $\hat{S}$ may be useful for the explanation of accidental degenerations of $\hat{H}$ eigenvalues (at some determined values of the Hamiltonian parameters). At the present time this problem is still under consideration.

References

Infeld L and Hull T E 1951 *Rev. Mod. Phys.* **23** 21

Landau L D and Lifshitz E M 1974 *Quantum Mechanics* (Moscow: Nauka) ch 2 (in Russian)

Miller W 1977 *Symmetry and Separation of Variables* (London: Addison-Wesley) ch 1

Ovsjannikov L V 1978 *Group Analysis of Differential Equations* (Moscow: Nauka) (in Russian)

Poluyanov L V and Voronin A I 1983 *J. Phys. A: Math. Gen.* **16** 3409

Pron'ko G P and Stroganov Yu G 1977 *Sov. Phys.-JETP* **72** 2048

Shapovalov V N 1969 *Siberian Math. J.* **10** 716

Wigner E P 1959 *Group Theory* (New York: Academic) ch 12